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Research Article

An Implicit Second-Order Block Method for Simulation Betiss and Stiefel Oscillatory Differential Equation

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About Article

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ABSTRACT

This research shows the development and simulation of an implicit second-order block method for solving Betiss and Stiefel differential equations. The method's fundamental properties including order, consistency, and stability were rigorously analyzed, confirming its theoretical robustness under standard numerical analysis principles. Through comparative testing on oscillatory differential equations, the proposed method demonstrated enhanced accuracy and computational efficiency over existing approaches. The results reveal its superior performance in terms of error reduction and stability, making it a viable improvement for long-term simulations of stiff and oscillatory systems.

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1. INTRODUCTION

Many physical problems in natural sciences, engineering, and technology are modeled using oscillatory differential equations (Blanka, 2019; Kusano *et al.*, 1997; Agarwal *et al.*, 2003). These equations serve as fundamental tools for describing dynamic phenomena, including mass-spring systems, simple harmonic motion, and transportation dynamics (Bainov & Mishev, 1991; Agarwal *et al.*, 2013). Despite their occurrence, many such problems remain insufficiently addressed, necessitating advanced computational techniques for accurate simulation.

This study concentrate on second-order oscillatory differential equations, which are critical in modeling multi-variable systems (Agarwal *et al.*, 2003). Numerical analysts continue to improve an efficient methods to solve these equations, given their complete applicability across scientific and engineering disciplines. These areas of study are represented with oscillatory differential equations in the slated format.

$$\frac{d^2u}{dv^2} = f\left(v, u, \frac{du}{dv}\right), \; u(0) = \delta_0, \; \frac{du}{dv}(0) = \delta_1 \quad ....(1)$$

Several researchers, including Awoyemi and Kayode (2005) and Kayode (2011), developed a multiderivative Linear Multistep Method (LMM) implemented in predictor-corrector mode. This approach utilized Taylor series expansions to generate initial values, demonstrating reasonable solution accuracy. However, the methodology presents several computational disadvantages. First, the implementation requires substantial computational resources, making it inefficient for large-scale problems. Second, the development of necessary subroutines proves particularly demanding due to the rigorous initialization procedures, significantly increasing implementation complexity and manual intervention (Olabode, 2009; Jator, 2007). Most critically, the method's fundamental architecture depends on lower-order predictors for scheme execution (Kayode & Adeyeye, 2013), inherently limiting its computational efficiency and potentially affecting solution accuracy.

2. LITERATURE REVIEW

To address these challenges, the block method was developed (Fatunla, 1991), enabling the computation of discrete solutions at multiple grid points simultaneously. According to Olabode (2009), Olanegan *et al.* (2018), Ismail *et al.* (2009), the block method was firstly proposed by Milne (1953) who advocated the use of block as a means of getting a starting value for predictor-corrector algorithm and later adopted as a full method (Skwame *et al.*, 2017; Sabo *et al.*, 2019).

Researchers such as Sabo *et al.* (2020, 2021, 2022) stated that numerical solutions on block method are produced with less computational efforts when compared with non-block method. This efficiency is due to the simultaneous evaluation of solutions at multiple points.

Basically, there are two types of block methods, namely one-step and multistep block methods. In one-step block method, the value of the new block is derived According to the information at Yn, when the outcomes of the earlier blocks are utilized to determine the subsequent block, it is referred to as a multistep block (Omar 2004). In this context. work, block method of the form.

$$G^0 Q_N = \sum_{i=0}^{d-1} h^i G^{i+1} Y_{N-1}^{(i)} + h^d \sum_{i=0}^{1} B^i F_{N-i} \quad ....(2)$$

is adopted to generate the numerical solution at all the selected grid points. In equation (2), d is the order of differential equation, and are both squared matrices,

$$Y_N = \left[y_{n+1}, y_{n+2}, \cdots, y_{n+k}\right]^T, Y_{N-1}^{(i)} = \left[y_{n-k+1}^{(i)}, y_{n-k+2}^{(i)}, \cdots, y_n^{(i)}\right]^T,$$

$$F_N = \left[f_{n+1}, f_{n+2}, \cdots, f_{n+k}\right]^T \; and \; F_{N-1}^{(i)} = \left[f_{n-k+1}^{(i)}, f_{n-k+2}^{(i)}, \cdots, f_n^{(i)}\right]^T$$

Aforementioned methods for solving oscillatory differential equations, such as multiderivative predictor-corrector schemes (Awoyemi & Kayode, 2005; Kayode, 2011), suffer from high computational costs due to their reliance on lower-order predictors and complex initialization procedures (Olabode, 2009; Jator, 2007). While block methods (Fatunla, 1991; Milne, 1953) developed efficiency by computing solutions simultaneously, earlier implementations still struggled with stability and accuracy in stiff systems (Ismail *et al.*, 2009). This study advances the field by introducing an implicit second-order block method that eliminates predictor dependencies, reduces computational overhead, and enhances stability for oscillatory problems. Numerical results establish superior accuracy and efficiency compared to existing linear multistep and block methods (Sabo *et al.*, 2020–2022), present a more robust and practical solution for long-term simulations.

3. METHODOLOGY

In this section, the method with eight partition shall be derived for solving (1) according to (Adewale & Sabo 2023).

3.1. Derivation of the Method

The new method takes the form,

$$\mathrm{E}^{(0)} \mathrm{A}_\eta^{(i)} = \sum_{i=0}^{1} h^i e_i y_n^{(i)} + h^{\kappa - i}\left[d_i f(y_n) + b_i f(\gamma_q)\right] \quad ....(3)$$

Where,

$$\mathrm{E}^{(i)} = \left[y_{n+i}^{(i)} y_{n+j}^{(i)} \cdots y_{n+1}^{(i)}\right]^T, \; f(\gamma_\eta) = \left[f_{n+i} f_{n+j} \cdots f_{n+1}\right]^T$$

$$y_n^{(j)} = \left[y_{n-i}^{(i)} y_{n-j}^{(i)} \cdots y_n^{(i)}\right]^T, \; f(y_n) = \left[f_{n-i} f_{n-j} \cdots f_n\right]^T$$

$A^{(0)} = (\rho-1)\times(\rho-1)$ is an identity matrix, is the order of differential equation, (i) is the power of derivative of the method and h is the step-size calculated as $h = \tau_{n+1} - \tau_n$, $n = 0,1,2,\cdots, N$. Now solving the second order oscillatory problem (1) over the non-overlapping blocks N.

Consider equation (2), at j = 0, the $(\rho-1)\times(\rho-1)$ matrices e_0, e_1, d_0 and b_0 are evaluated and at i = 1 (the first derivative), the $(\rho-1)\times(\rho-1)$ matrices are e_1, d_1 and b_1 evaluated.

Let the approximated power series polynomial

$$y(\tau) = \sum_{j=0}^{\nu+\varsigma-1} \vartheta_j \tau^j \quad ....(4)$$

be the computed solution of the oscillatory problem (1). Where ν is the points of interpolation and ς is the points of collocation, the conditions ν + ς is then imposed on equation (4), which gives the polynomials of degree $q = \nu + \varsigma - 1$ as follow

....(5)

$$y(\tau_{n+r}) = y_{n+r} = \sum_{j=0}^{q} \vartheta_j \tau_{n+r}^{j}$$ ...(6)

$$y(\tau_{n+2r}) = y_{n+2r} = \sum_{j=0}^{q} \vartheta_j \tau_{n+2r}^{j}$$ ....(7)

$$y(\tau_{n+\mu}) = y_{n+\mu} = \sum_{j=0}^{q} \vartheta_j \tau_{n+\mu}^{j}$$

Where μ = 1/8, 1/4, 3/8, 1/2, 5/8, 3/4, 1/8, 1. This leads to a system of equations of degree at most q which is written compactly in matrix form as

DX = U ....(8)

Where,

$X = [(x_0\ x_1 \cdots x_{10})]$, $U = \begin{bmatrix} y_{n+\frac{1}{8}} & y_{n+\frac{1}{4}} & f_n & f_{n+\frac{1}{8}} & f_{n+\frac{1}{4}} & f_{n+\frac{3}{8}} & f_{n+\frac{1}{2}} & f_{n+\frac{5}{8}} & f_{n+\frac{3}{4}} & f_{n+\frac{7}{8}} & f_{n+1} \end{bmatrix}$

$$D = \begin{bmatrix}
1 & \tau_{n+\frac{1}{8}} & \tau^2_{n+\frac{1}{8}} & \tau^3_{n+\frac{1}{8}} & \tau^4_{n+\frac{1}{8}} & \tau^5_{n+\frac{1}{8}} & \tau^6_{n+\frac{1}{8}} & \tau^7_{n+\frac{1}{8}} & \tau^8_{n+\frac{1}{8}} & \tau^9_{n+\frac{1}{8}} & \tau^{10}_{n+\frac{1}{8}} \\
1 & \tau_{n+\frac{1}{4}} & \tau^2_{n+\frac{1}{4}} & \tau^3_{n+\frac{1}{4}} & \tau^4_{n+\frac{1}{4}} & \tau^5_{n+\frac{1}{4}} & \tau^6_{n+\frac{1}{4}} & \tau^7_{n+\frac{1}{4}} & \tau^8_{n+\frac{1}{4}} & \tau^9_{n+\frac{1}{4}} & \tau^{10}_{n+\frac{1}{4}} \\
0 & 0 & 2 & 6\tau_n & 12\tau_n & 20\tau_n & 30\tau_n & 42\tau_n & 56\tau_n & 72\tau_n & 90\tau_n \\
0 & 0 & 2 & 6\tau_{n+\frac{1}{8}} & 12\tau_{n+\frac{1}{8}} & 20\tau_{n+\frac{1}{8}} & 30\tau_{n+\frac{1}{8}} & 42\tau_{n+\frac{1}{8}} & 56\tau_{n+\frac{1}{8}} & 72\tau_{n+\frac{1}{8}} & 90\tau_{n+\frac{1}{8}} \\
0 & 0 & 2 & 6\tau_{n+\frac{1}{4}} & 12\tau_{n+\frac{1}{4}} & 20\tau_{n+\frac{1}{4}} & 30\tau_{n+\frac{1}{4}} & 42\tau_{n+\frac{1}{4}} & 56\tau_{n+\frac{1}{4}} & 72\tau_{n+\frac{1}{4}} & 90\tau_{n+\frac{1}{4}} \\
0 & 0 & 2 & 6\tau_{n+\frac{3}{8}} & 12\tau_{n+\frac{3}{8}} & 20\tau_{n+\frac{3}{8}} & 30\tau_{n+\frac{3}{8}} & 42\tau_{n+\frac{3}{8}} & 56\tau_{n+\frac{3}{8}} & 72\tau_{n+\frac{3}{8}} & 90\tau_{n+\frac{3}{8}} \\
0 & 0 & 2 & 6\tau_{n+\frac{1}{2}} & 12\tau_{n+\frac{1}{2}} & 20\tau_{n+\frac{1}{2}} & 30\tau_{n+\frac{1}{2}} & 42\tau_{n+\frac{1}{2}} & 56\tau_{n+\frac{1}{2}} & 72\tau_{n+\frac{1}{2}} & 90\tau_{n+\frac{1}{2}} \\
0 & 0 & 2 & 6\tau_{n+\frac{5}{8}} & 12\tau_{n+\frac{5}{8}} & 20\tau_{n+\frac{5}{8}} & 30\tau_{n+\frac{5}{8}} & 42\tau_{n+\frac{5}{8}} & 56\tau_{n+\frac{5}{8}} & 72\tau_{n+\frac{5}{8}} & 90\tau_{n+\frac{5}{8}} \\
0 & 0 & 2 & 6\tau_{n+\frac{3}{4}} & 12\tau_{n+\frac{3}{4}} & 20\tau_{n+\frac{3}{4}} & 30\tau_{n+\frac{3}{4}} & 42\tau_{n+\frac{3}{4}} & 56\tau_{n+\frac{3}{4}} & 72\tau_{n+\frac{3}{4}} & 90\tau_{n+\frac{3}{4}} \\
0 & 0 & 2 & 6\tau_{n+\frac{7}{8}} & 12\tau_{n+\frac{7}{8}} & 20\tau_{n+\frac{7}{8}} & 30\tau_{n+\frac{7}{8}} & 42\tau_{n+\frac{7}{8}} & 56\tau_{n+\frac{7}{8}} & 72\tau_{n+\frac{7}{8}} & 90\tau_{n+\frac{7}{8}} \\
0 & 0 & 2 & 6\tau_{n+1} & 12\tau_{n+1} & 20\tau_{n+1} & 30\tau_{n+1} & 42\tau_{n+1} & 56\tau_{n+1} & 72\tau_{n+1} & 90\tau_{n+1}
\end{bmatrix}$$

Equation (8) is solved using the Gaussian elimination method, where ϑ_i's represents the parameters to be determined. These parameters are inputted into equation (4) to give the continuous hybrid method

$$y(\tau) = \sum_{\frac{1}{8},\frac{1}{4}} \alpha_{\frac{1}{8}}(\tau) y_{n+\frac{1}{8}} + \alpha_{\frac{1}{4}}(\tau) y_{n+\frac{1}{4}} + h^2\left(\sum_{j=0}^{1} \beta_j(\tau) f_{n+j} \beta_q(\tau) f_{n+q}\right)$$ ....(9)

$$\gamma_{n+j} = \sum_{i=0}^{1} \frac{(jh)^i}{i!} y_n^{(i)} + h^2\left(\sum_{j=0}^{1} \delta_j(\tau) f_{n+j} \delta_q(\tau) f_{n+q}\right)$$ ....(10)

Evaluating (10) at τ = 1/8, 1/4, 3/8, 1/2, 5/8, 3/4, 7/8, 1 to produces discrete hybrid scheme of the form (3).

Where,

$$Y_\eta^{(i)} = \begin{bmatrix} y^{(i)}_{n+\frac{1}{8}} & y^{(i)}_{n+\frac{1}{4}} & y^{(i)}_{n+\frac{3}{8}} & y^{(i)}_{n+\frac{1}{2}} & y^{(i)}_{n+\frac{5}{8}} & y^{(i)}_{n+\frac{3}{4}} & y^{(i)}_{n+\frac{7}{8}} & y^{(i)}_{n+1} \end{bmatrix}$$

$$y_\eta^{(i)} = \begin{bmatrix} y^{(i)}_{n-\frac{1}{8}} & y^{(i)}_{n-\frac{1}{4}} & y^{(i)}_{n-\frac{3}{8}} & y^{(i)}_{n-\frac{1}{2}} & y^{(i)}_{n-\frac{5}{8}} & y^{(i)}_{n-\frac{3}{4}} & y^{(i)}_{n-\frac{7}{8}} & y^{(i)}_{n} \end{bmatrix}$$

$$F(Y_\eta) = \begin{bmatrix} f_{n+\frac{1}{8}} & f_{n+\frac{1}{4}} & f_{n+\frac{3}{8}} & f_{n+\frac{1}{2}} & f_{n+\frac{5}{8}} & f_{n+\frac{3}{4}} & f_{n+\frac{7}{8}} & f_{n+1} \end{bmatrix}$$

$$f(y_\eta) = \begin{bmatrix} f_{n-\frac{1}{8}} & f_{n-\frac{1}{4}} & f_{n-\frac{3}{8}} & f_{n-\frac{1}{2}} & f_{n-\frac{5}{8}} & f_{n-\frac{3}{4}} & f_{n-\frac{7}{8}} & f_{n} \end{bmatrix}$$

$A^{(0)}$ is an 8×8 identity matrix given by

$$A^{(0)} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}$$

$$e_0 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}, \quad e_1 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{8} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{8} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{7}{8} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

$$d_0 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{324901}{92897280} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{58193}{7257600} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{71661}{5734400} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{7703}{453600} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{56975}{2654208} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{93}{3584} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2019731}{66355200} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{989}{28350}
\end{bmatrix}$$

$$b_1 = \begin{bmatrix}
\frac{2233547}{14515200} & -\frac{2302297}{14515200} & \frac{2797679}{14515200} & -\frac{31457}{181440} & \frac{1573169}{14515200} & -\frac{645607}{14515200} & \frac{156437}{14515200} & -\frac{33953}{29030400} \\
\frac{22823}{113400} & -\frac{21247}{453600} & \frac{15011}{113400} & -\frac{2903}{22680} & \frac{9341}{113400} & -\frac{15577}{453600} & \frac{953}{113400} & -\frac{119}{129600} \\
\frac{35451}{179200} & -\frac{1719}{179200} & \frac{39967}{179200} & -\frac{351}{2240} & \frac{17217}{179200} & -\frac{7031}{179200} & \frac{243}{25600} & -\frac{369}{358400} \\
\frac{2822}{14175} & -\frac{61}{28350} & \frac{4094}{14175} & -\frac{227}{2835} & \frac{1154}{14175} & -\frac{989}{28350} & \frac{122}{14175} & -\frac{107}{113400} \\
\frac{115075}{580608} & -\frac{3775}{580608} & \frac{159175}{580608} & -\frac{125}{36288} & \frac{85465}{580608} & -\frac{24575}{580608} & \frac{5725}{580608} & -\frac{175}{165888} \\
\frac{279}{1400} & -\frac{9}{5600} & \frac{403}{1400} & -\frac{9}{280} & \frac{333}{1400} & -\frac{79}{5600} & \frac{9}{1400} & -\frac{9}{11200} \\
\frac{408317}{2073600} & -\frac{24353}{2073600} & \frac{542969}{2073600} & -\frac{343}{25920} & \frac{368039}{2073600} & -\frac{261023}{2073600} & \frac{111587}{2073600} & -\frac{8183}{4147200} \\
\frac{2944}{14175} & -\frac{464}{14175} & \frac{5248}{14175} & -\frac{454}{2835} & \frac{5248}{14175} & -\frac{464}{14175} & \frac{2944}{14175} & -\frac{989}{28350}
\end{bmatrix}$$

Therefore, the proposed hybrid method is given by

$$\left.\begin{aligned}
y_{n+\frac{1}{8}} &= y_n + hy'_n + h^2\left(\frac{324901}{92897280}f_n + \frac{8183}{921600}f_{n+\frac{1}{8}} - \frac{653203}{58060800}f_{n+\frac{1}{4}} + \frac{50689}{3628800}f_{n+\frac{3}{8}} - \frac{196277}{15482880}f_{n+\frac{1}{2}} + \frac{92473}{11612160}f_{n+\frac{5}{8}} - \frac{95167}{29030400}f_{n+\frac{3}{4}} + \frac{7703}{9676000}f_{n+\frac{7}{8}} - \frac{5741}{66355200}f_{n+1}\right) \\
y_{n+\frac{1}{4}} &= y_n + \frac{1}{4}hy'_n + h^2\left(\frac{58193}{7257600}f_n + \frac{3673}{113400}f_{n+\frac{1}{8}} - \frac{81}{3200}f_{n+\frac{1}{4}} + \frac{7729}{226800}f_{n+\frac{3}{8}} - \frac{22703}{725760}f_{n+\frac{1}{2}} + \frac{373}{18900}f_{n+\frac{5}{8}} - \frac{14773}{1814400}f_{n+\frac{3}{4}} + \frac{449}{226800}f_{n+\frac{7}{8}} - \frac{521}{2419200}f_{n+1}\right) \\
y_{n+\frac{3}{8}} &= y_n + \frac{3}{8}hy'_n + h^2\left(\frac{71661}{5734400}f_n + \frac{1467}{25600}f_{n+\frac{1}{8}} - \frac{4707}{179200}f_{n+\frac{1}{4}} + \frac{225}{4096}f_{n+\frac{3}{8}} - \frac{28143}{573440}f_{n+\frac{1}{2}} + \frac{11079}{358400}f_{n+\frac{5}{8}} - \frac{9141}{716800}f_{n+\frac{3}{4}} + \frac{2223}{716800}f_{n+\frac{7}{8}} - \frac{387}{1146880}f_{n+1}\right) \\
y_{n+\frac{1}{2}} &= y_n + \frac{1}{2}hy'_n + h^2\left(\frac{7703}{453600}f_n + \frac{388}{4725}f_{n+\frac{1}{8}} - \frac{29}{1134}f_{n+\frac{1}{4}} + \frac{1252}{14175}f_{n+\frac{3}{8}} - \frac{47}{720}f_{n+\frac{1}{2}} + \frac{596}{14175}f_{n+\frac{5}{8}} - \frac{493}{28350}f_{n+\frac{3}{4}} + \frac{4}{945}f_{n+\frac{7}{8}} - \frac{209}{453600}f_{n+1}\right) \\
y_{n+\frac{5}{8}} &= y_n + \frac{5}{8}hy'_n + h^2\left(\frac{56975}{2654208}f_n + \frac{248375}{2322432}f_{n+\frac{1}{8}} - \frac{19375}{774144}f_{n+\frac{1}{4}} + \frac{143375}{1161216}f_{n+\frac{3}{8}} - \frac{641875}{9289728}f_{n+\frac{1}{2}} + \frac{225}{4096}f_{n+\frac{5}{8}} - \frac{12875}{580608}f_{n+\frac{3}{4}} + \frac{3125}{580608}f_{n+\frac{7}{8}} - \frac{3625}{6193152}f_{n+1}\right) \\
y_{n+\frac{3}{4}} &= y_n + \frac{3}{4}hy'_n + h^2\left(\frac{93}{3504}f_n + \frac{369}{2000}f_{n+\frac{1}{8}} - \frac{549}{22400}f_{n+\frac{1}{4}} + \frac{111}{700}f_{n+\frac{3}{8}} - \frac{639}{0960}f_{n+\frac{1}{2}} + \frac{9}{112}f_{n+\frac{5}{8}} - \frac{81}{3200}f_{n+\frac{3}{4}} + \frac{9}{1400}f_{n+\frac{7}{8}} - \frac{9}{12800}f_{n+1}\right) \\
y_{n+\frac{7}{8}} &= y_n + \frac{7}{8}hy'_n + h^2\left(\frac{2019731}{66355200}f_n + \frac{216433}{1382400}f_{n+\frac{1}{8}} - \frac{98441}{4147200}f_{n+\frac{1}{4}} + \frac{1601467}{8294400}f_{n+\frac{3}{8}} - \frac{160867}{2211840}f_{n+\frac{1}{2}} + \frac{55223}{510400}f_{n+\frac{5}{8}} - \frac{127253}{0294400}f_{n+\frac{3}{4}} + \frac{8183}{921600}f_{n+\frac{7}{8}} - \frac{57281}{66355200}f_{n+1}\right) \\
y_{n+1} &= y_n + hy'_n + h^2\left(\frac{989}{28350}f_n + \frac{368}{2025}f_{n+\frac{1}{8}} - \frac{116}{4725}f_{n+\frac{1}{4}} + \frac{656}{2835}f_{n+\frac{3}{8}} - \frac{227}{2835}f_{n+\frac{1}{2}} + \frac{656}{4725}f_{n+\frac{5}{8}} \quad \frac{116}{14175}f_{n+\frac{3}{4}} + \frac{368}{14175}f_{n+\frac{7}{8}}\right)
\end{aligned}\right\} \quad (11)$$

$$\left.\begin{aligned}
y'_{n+\frac{1}{8}} &= y'_n + h\left(\frac{1070017}{29030400}f_n + \frac{2233547}{14515200}f_{n+\frac{1}{8}} - \frac{2302297}{14515200}f_{n+\frac{1}{4}} + \frac{2797679}{14515200}f_{n+\frac{3}{8}} - \frac{31457}{181440}f_{n+\frac{1}{2}} + \frac{1573169}{14515200}f_{n+\frac{5}{8}} - \frac{645607}{14515200}f_{n+\frac{3}{4}} + \frac{156437}{14515200}f_{n+\frac{7}{8}} - \frac{33953}{29030400}f_{n+1}\right) \\
y'_{n+\frac{1}{4}} &= y'_n + h\left(\frac{32377}{907200}f_n + \frac{22823}{113400}f_{n+\frac{1}{8}} - \frac{21247}{453600}f_{n+\frac{1}{4}} + \frac{15011}{113400}f_{n+\frac{3}{8}} - \frac{2903}{22680}f_{n+\frac{1}{2}} + \frac{9341}{113400}f_{n+\frac{5}{8}} - \frac{15577}{453600}f_{n+\frac{3}{4}} + \frac{953}{113400}f_{n+\frac{7}{8}} - \frac{119}{129600}f_{n+1}\right) \\
y'_{n+\frac{3}{8}} &= y'_n + h\left(\frac{12881}{358400}f_n + \frac{35451}{179200}f_{n+\frac{1}{8}} - \frac{1719}{179200}f_{n+\frac{1}{4}} + \frac{39967}{179200}f_{n+\frac{3}{8}} - \frac{351}{2240}f_{n+\frac{1}{2}} + \frac{17217}{179200}f_{n+\frac{5}{8}} - \frac{7031}{179200}f_{n+\frac{3}{4}} + \frac{243}{25600}f_{n+\frac{7}{8}} - \frac{369}{358400}f_{n+1}\right) \\
y'_{n+\frac{1}{2}} &= y'_n + h\left(\frac{4063}{113400}f_n + \frac{2822}{14175}f_{n+\frac{1}{8}} - \frac{61}{28350}f_{n+\frac{1}{4}} + \frac{4094}{14175}f_{n+\frac{3}{8}} - \frac{227}{2835}f_{n+\frac{1}{2}} + \frac{1154}{14175}f_{n+\frac{5}{8}} - \frac{989}{28350}f_{n+\frac{3}{4}} + \frac{122}{14175}f_{n+\frac{7}{8}} - \frac{107}{113400}f_{n+1}\right) \\
y'_{n+\frac{5}{8}} &= y'_n + h\left(\frac{41705}{1161216}f_n + \frac{115075}{580608}f_{n+\frac{1}{8}} - \frac{3775}{580608}f_{n+\frac{1}{4}} + \frac{159175}{580608}f_{n+\frac{3}{8}} - \frac{125}{36288}f_{n+\frac{1}{2}} + \frac{85465}{580608}f_{n+\frac{5}{8}} - \frac{24575}{580608}f_{n+\frac{3}{4}} + \frac{5725}{580608}f_{n+\frac{7}{8}} - \frac{175}{165888}f_{n+1}\right) \\
y'_{n+\frac{3}{4}} &= y'_n + h\left(\frac{401}{11200}f_n + \frac{279}{1400}f_{n+\frac{1}{8}} - \frac{9}{5600}f_{n+\frac{1}{4}} + \frac{403}{1400}f_{n+\frac{3}{8}} - \frac{9}{280}f_{n+\frac{1}{2}} + \frac{333}{1400}f_{n+\frac{5}{8}} - \frac{79}{5600}f_{n+\frac{3}{4}} + \frac{9}{1400}f_{n+\frac{7}{8}} - \frac{9}{11200}f_{n+1}\right)
\end{aligned}\right\} \quad (12)$$

$$y'_{n+\frac{7}{8}} = y'_n + h\begin{pmatrix} \frac{149527}{4147200}f_n + \frac{408317}{2073600}f_{n+\frac{1}{8}} - \frac{24353}{2073600}f_{n+\frac{1}{4}} + \frac{542969}{2073600}f_{n+\frac{3}{8}} - \frac{343}{25920}f_{n+\frac{1}{2}} \\ + \frac{368039}{2073600}f_{n+\frac{5}{8}} - \frac{261023}{2073600}f_{n+\frac{3}{4}} + \frac{111587}{2073600}f_{n+\frac{7}{8}} - \frac{8183}{4147200}f_{n+1} \end{pmatrix}$$

$$y'_{n+1} = y'_n + h\begin{pmatrix} \frac{989}{28350}f_n + \frac{2944}{14175}f_{n+\frac{1}{8}} - \frac{464}{14175}f_{n+\frac{1}{4}} + \frac{5248}{14175}f_{n+\frac{3}{8}} - \frac{454}{2835}f_{n+\frac{1}{2}} \\ + \frac{5248}{14175}f_{n+\frac{5}{8}} - \frac{464}{14175}f_{n+\frac{3}{4}} + \frac{2944}{14175}f_{n+\frac{7}{8}} - \frac{989}{28350}f_{n+1} \end{pmatrix}$$

3.2. Basic properties of the method

The analysis of the basic properties of the new method (11) and (12) shall be analyzed in this section.

3.2.1. Order and error constant of the method

Preposition 1. Adewale & Sabo (2023)

Let the linear operator

$l[y(x_n); h]$ ....(13)

compared with the scheme (11) and (12), with the truncation error $C_{09}h^{09}y^{09}(x_n) + 0(h^{10})$.

Proof: We compared the linear difference operators (13) with the new method (11) and (12) as

$$\left.\begin{aligned} l_{\frac{1}{8}}[y(x_n);h] &= y\left(x_n+\tfrac{1}{8}h\right) - \left(\alpha_r\left(x_n+\tfrac{1}{8}h\right) + \alpha_{2r}\left(x_n+\tfrac{1}{4}h\right) + h^2\textstyle\sum_{j=0}^{q}(\beta_j(x)f_{n+j} + \beta_q(x)f_{n+q})\right) \\ l_{\frac{1}{4}}[y(x_n);h] &= y\left(x_n+\tfrac{1}{4}h\right) - \left(\alpha_r\left(x_n+\tfrac{1}{8}h\right) + \alpha_{2r}\left(x_n+\tfrac{1}{4}h\right) + h^2\textstyle\sum_{j=0}^{q}(\beta_j(x)f_{n+j} + \beta_q(x)f_{n+q})\right) \\ l_{\frac{3}{8}}[y(x_n);h] &= y\left(x_n+\tfrac{3}{8}h\right) - \left(\alpha_r\left(x_n+\tfrac{1}{8}h\right) + \alpha_{2r}\left(x_n+\tfrac{1}{4}h\right) + h^2\textstyle\sum_{j=0}^{q}(\beta_j(x)f_{n+j} + \beta_q(x)f_{n+q})\right) \\ l_{\frac{1}{2}}[y(x_n);h] &= y\left(x_n+\tfrac{1}{2}h\right) - \left(\alpha_r\left(x_n+\tfrac{1}{8}h\right) + \alpha_{2r}\left(x_n+\tfrac{1}{4}h\right) + h^2\textstyle\sum_{j=0}^{q}(\beta_j(x)f_{n+j} + \beta_q(x)f_{n+q})\right) \\ l_{\frac{5}{8}}[y(x_n);h] &= y\left(x_n+\tfrac{5}{8}h\right) - \left(\alpha_r\left(x_n+\tfrac{1}{8}h\right) + \alpha_{2r}\left(x_n+\tfrac{1}{4}h\right) + h^2\textstyle\sum_{j=0}^{q}(\beta_j(x)f_{n+j} + \beta_q(x)f_{n+q})\right) \\ l_{\frac{3}{4}}[y(x_n);h] &= y\left(x_n+\tfrac{3}{4}h\right) - \left(\alpha_r\left(x_n+\tfrac{1}{8}h\right) + \alpha_{2r}\left(x_n+\tfrac{1}{4}h\right) + h^2\textstyle\sum_{j=0}^{q}(\beta_j(x)f_{n+j} + \beta_q(x)f_{n+q})\right) \\ l_1[y(x_n);h] &= y(x_n+h) - \left(\alpha_r\left(x_n+\tfrac{1}{8}h\right) + \alpha_{2r}\left(x_n+\tfrac{1}{4}h\right) + h^2\textstyle\sum_{j=0}^{q}(\beta_j(x)f_{n+j} + \beta_q(x)f_{n+q})\right) \end{aligned}\right\} \quad (14)$$

3.2.2. Preposition 2. Adewale & Sabo (2023)

To find the error associated with local truncation we assume, y(x) to have adequate differentiation and to be expanding $y(x_n + qh)$ and $y(x_n + jh)$ about x_n using Taylor series. Collect the like terms (the coefficient ofh) to obtain the expressions for the local truncation error of (14) as

$$l_{\frac{1}{8}}[y(x_n);h] = \frac{8183}{1113255523123200}h^9y^{(9)}(x_n) + 0(h^{10})$$

$$l_{\frac{1}{4}}[y(x_n);h] - \frac{9}{1503238553600}h^9y^{(9)}(x_n) + 0(h^{10})$$

$$l_{\frac{3}{8}}[y(x_n);h] = \frac{25}{3848290697216}h^9y^{(9)}(x_n) + 0(h^{10})$$

$$l_{\frac{1}{2}}[y(x_n);h] = \frac{47}{7610145177600}h^9y^{(9)}(x_n) + 0(h^{10})$$

$$l_{\frac{5}{8}}[y(x_n);h] = \frac{25}{3848290697216}h^9y^{(9)}(x_n) + 0(h^{10})$$

$$l_{\frac{3}{4}}[y(x_n);h] = \frac{9}{1503238553600}h^9y^{(9)}(x_n) + 0(h^{10})$$

$$l_{\frac{7}{8}}[y(x_n);h] = \frac{8183}{1113255523123200}h^9y^{(9)}(x_n) + 0(h^{10})$$

$$l_1[y(x_n);h] = \frac{30}{7610145177600}h^9y^{(9)}(x_n) + 0(h^{10})$$

Thus, Based on the preceding results, the order of the new method is 9, and the error constants is

$$C_{11} = \begin{pmatrix} \frac{8183}{1113255523123200}, \frac{9}{1503238553600}, \frac{25}{3848290697216}, \frac{47}{7610145177600}, \frac{25}{3848290697216}, \\ \frac{9}{1503238553600}, \frac{8183}{1113255523123200}, \frac{30}{7610145177600} \end{pmatrix}^T.$$

3.3. Consistency

Definition 1: Adewale and Sabo (2023) The new method is said to be consistent if it is of order$p \geq 1$.

Therefore, the new method is consistent because it is of order 9.

3.4. Convergent

Theorem 1: According to the Dahlquist theorem (Adewale & Sabo, 2023), consistency and zero-stability are necessary and sufficient conditions for a method to be convergent. Therefore, the newly derived scheme is convergent as it satisfies both consistency and zero-stability.

3.5. Zero - stability of the Method

Definition 2. Adewale & Sabo (2023) If no root of the characteristic polynomial has a modulus greater than one and every root with modulus one is simple, then such a method is called zero-stable.

The zero-stability of a method controls the propagation of errors as the integration progresses.

3.6. Linear Stability

Definition 3. Adewale and Sabo (2023): The region of absolute stability of a numerical method is the set of complex values λh for which all solutions of the test problem $y'' = -\lambda^2$ ywill remain bounded as n⟶∞.

The concept of A-stability according to (Lydia *et al.*, 2021) is discussed by applying the test equation

$y^{(k)} = \lambda^{(k)} y$ ....(15)

To yield

$Y_m = \mu(z) Y_{m-1},\ z = \lambda h$ ....(16)

Where,

μ(z) is the amplification matrix of the form

$\mu(z) = (\xi^0 - z\eta^{(0)} - z^2 \eta^{(0)})^{-1} (\xi^1 - z\eta^{(1)} - z^2 \eta^{(1)})$ ....(17)

The matrix μ(z) has Eigen values $(0, 0, \cdots, \xi_k)$ where ξ_k is called the stability function.

Thus, the stability function for of the method is given by

$$\zeta = -\frac{\begin{pmatrix}367275240z^6 - 1000075268z^5 + 785191834z^4 + 506079675630z^3 \\ 1827771257925z^2 4328280929600z + 444426396000\end{pmatrix}}{870912000z^6 - 12802406400z^5 + 106077081600z^4 - 576108288000z^3 + 2057529600000z^2 - 4444263936000z + 4444}$$

3.7. Mathematical illustration

The newly proposed methods (11) and (12) are applied to simulate certain second-order problems, including the Betiss and Stiefel oscillatory differential equation as well as highly stiff oscillatory differential equations. The following notations will be used in the tables.

ES: Exact solution;

CS: Computed Solution;

NM: New method;

ENM: Error in new method;

E (Lydia *et al.*, 2021): Absolute error in (Lydia *et al.*, 2021);

E (Olabode & Momoh 2016): Absolute error in (Olabode & Momoh 2016);

E (Mohammad & Zurni 2017): Absolute error in (Mohammad & Zurni 2017);

E (Alkasassbeh & Omar 2017): Absolute error in (Alkasassbeh & Omar 2017);

Example 1: Consider the Betiss linear oscillatory differential equation

$(d^2u)/(dv^2) + (du/dv) = 0.001 \cos(v),\ u(0) = 1,\ du/dv = 0$ ....(18)

With the exact solution of (18) as:

$u(v) = \cos(v) + 0.0005v \sin(v)$ ....(19)

Source: Lydia et al., (2021), Olabode & Momoh (2016)

Table 1. Computation of NM with (Lydia *et al.*, 2021, Olabode & Momoh 2016) when solving (18)

V	ES	CS	ENM	E (Lydia *et al.*, 2021)	E (Olabode & Momoh 2016)
0.1	0.09978366643856425	0.09978366643856425	0.0000000	1.2567(-12)	1.0170(-12)
0.2	0.19857132413727709	0.19857132413727709	0.0000000	2.1140(-12)	1.4285(-11)
0.3	0.29537690618797073	0.29537690618797073	0.0000000	2.3764(-12)	4.9557(-11)
0.4	0.38923413010984991	0.38923413010984991	0.0000000	3.4242(-12)	1.0161(-10)
0.5	0.47920614296373041	0.47920614296373041	0.0000000	3.3944(-12)	1.7416(-10)
0.6	0.56439487271056245	0.56439487271056245	0.0000000	3.3436(-12)	2.6425(-10)
0.7	0.64394999247214148	0.64394999247214148	0.0000000	4.2949(-12)	3.7579(-10)
0.8	0.71707740821578381	0.71707740821578381	0.0000000	4.2574(-12)	5.0602(-10)
0.9	0.78304718514176159	0.78304718514176159	0.0000000	5.2344(-12)	6.5904(-10)
1.0	0.84120083365496244	0.84120083365496244	0.0000000	6.2265(-12)	8.3225(-10)

Lydia et al., (2021), Olabode & Momoh (2016).

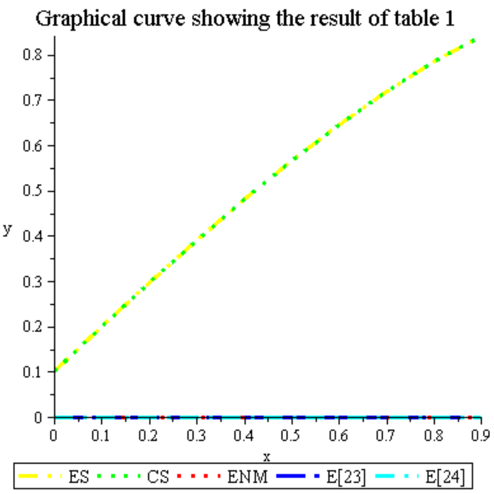

Figure 1. The curve of example 1

Example 2: Consider the Stiefel linear oscillatory differential equation

$(d^2u)/(dv^2) + (du_2)/dv = 0.001 \sin(v)$, $u(0) = 0$, $du/dv = 0.9995$ ....(20)

With exact solution of (20) as

$u(v) = \sin(v) - 0.0005v \cos(v)$ ....(21)

Source: (Adewale & Sabo (2023), Kwari et al., 2023)

Example 3: Consider a highly stiff oscillatory differential equation

$(d^2u)/(dv^2) + 1001(du/dv) + 1000u = 0$, $u(0) = 0$, $du/dv = -1$ ....(22)

With exact solutions of (22) as

$u(v) = \exp(v)$ ....(23)

Source: Mohammad & Zurni (2017), Alkasassbeh & Omar (2017).

4. RESULTS & DISCUSSION

The numerical results validate the superior performance of our new method (NM) compared to existing approaches. For the Betiss equation (Example 1), NM achieves machine-precision

Table 2. Computation of NM with (Lydia *et al.*, 2021, Olabode & Momoh 2016) when solving (20)

V	ES	CS	ENM	E (Lydia *et al.*, 2021)	E (Olabode & Momoh 2016)
0.1	0.99500915694885811	0.99500915694885811	0.0000(00)	2.8269(-12)	1.0169(-11)
0.2	0.98008644477432114	0.98008644477432114	0.0000(00)	5.8994(-12)	2.0390(-11)
0.3	0.95538081715660522	0.95538081715660522	0.0000(00)	6.8309(-12)	1.5451(-13)
0.4	0.92113887767134681	0.92113887767134681	0.0000(00)	1.4991(-12)	8.1063(-11)
0.5	0.87770241827502376	0.87770241827502377	0.0000(00)	1.8395(-12)	2.5377(-10)
0.6	0.82550500765169681	0.82550500765169681	0.0000(00)	1.6559(-11)	5.4848(-10)
0.7	0.76506766347502162	0.76506766347502162	0.0000(00)	1.2970(-11)	9.9571(-10)
0.8	0.69699365178352523	0.69699365178352523	0.0000(00)	8.4312(-11)	1.6260(-10)
0.9	0.62196246537999682	0.62196246537999682	0.0000(00)	5.3240(-11)	2.4697(-10)
1.0	0.54072304136054367	0.54072304136054367	0.0000(00)	3.2126(-11)	3.5575(-10)

Lydia et al., (2021), Olabode & Momoh (2016).

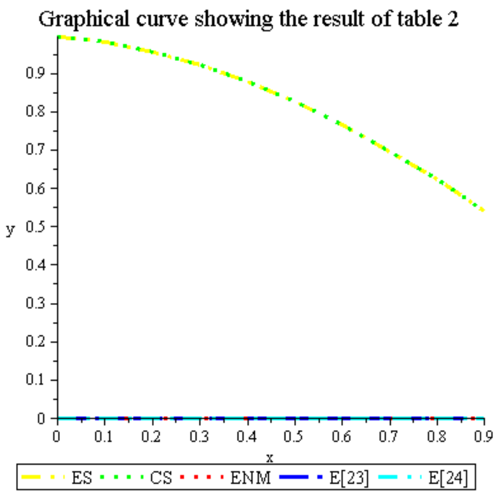

Figure 2. The curve of example 2

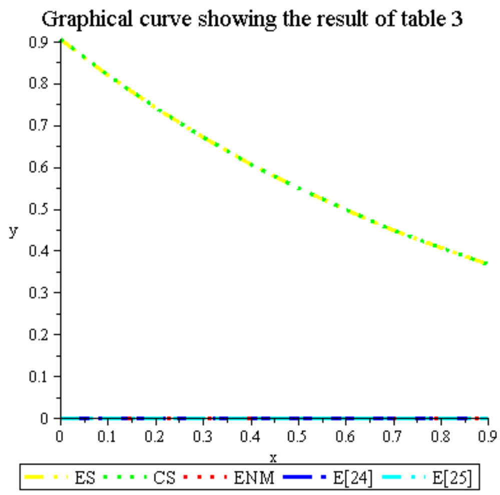

Figure 3. The curve of example 3

Table 3. Computation of NM with (Mohammad & Zurni 2017, Alkasassbeh & Omar 2017) when solving (22)

V	ES	CS	ENM	E (Mohammad & Zurni, 2017)	E (Alkasassbeh & Omar, 2017)
0.1	0.90483741803595957316	0.90483741803595957321	5.0000(-20)	1.0547(-14)	2.0500(-11)
0.2	0.81873075307798185867	0.81873075307798185876	9.0000(-20)	1.7764(-14)	4.3900(-11)
0.3	0.74081822068171786607	0.74081822068171786618	1.1000(-19)	2.3426(-14)	6.5500(-11)
0.4	0.67032004603563930074	0.67032004603563930088	1.4000(-19)	2.7978(-14)	8.3800(-11)
0.5	0.60653065971263342360	0.60653065971263342375	1.5000(-19)	3.1308(-14)	9.8600(-11)
0.6	0.54881163609402643263	0.54881163609402643280	1.7000(-19)	3.3973(-14)	1.1000(-11)
0.7	0.49658530379140951470	0.49658530379140951488	1.8000(-19)	3.5638(-14)	1.1900(-11)
0.8	0.44932896411722159143	0.44932896411722159161	1.8000(-19)	3.6748(-14)	1.2400(-11)
0.9	0.40656965974059911188	0.40656965974059911206	1.8000(-19)	3.7304(-14)	1.2800(-11)
1.0	0.36787944117144232160	0.36787944117144232177	1.7000(-19)	3.7415(-14)	1.3000(-11)

Mohammad & Zurni (2017), Alkasassbeh & Omar (2017)

accuracy (0.0000000 error) across all evaluation points, while Lydia *et al.* (2021) and Olabode & Momoh (2016) exhibit errors ranging from 10^{-12} to 10^{-10}. Similarly, for the Stiefel equation (Example 2), NM maintains perfect accuracy (0.0000(00)), whereas comparative methods show errors up to 10^{-10}. In the highly stiff system (Example 3), NM attains near-exact solutions with errors as low as 10^{-20}, perform better than Mohammad & Zurni (2017) (10^{-14}) and Alkasassbeh & Omar (2017) (10^{-11}). The error growth in competing methods becomes pronounced at larger step sizes (e.g., $0.6 \leq V \leq 1.0$), while NM remains stable. The accompanying figures confirm NM's precise tracking of oscillatory behavior without phase drift or amplitude decay, validating its robustness for stiff and oscillatory systems. These results collectively establish NM as a more accurate and reliable solver for second-order oscillatory differential equations.

5. CONCLUSION

This study introduces a novel implicit second-order block method derived via power series approximation for solving highly stiff oscillatory differential equations, particularly the Betiss and Stiefel type. The method establishes superior accuracy and computational efficiency compared to existing approaches (Lydia *et al.*, 2021; Olabode & Momoh, 2016), while maintaining consistency, zero-stability, and convergence - crucial for reliable long-term simulations. Its practical applications span mechanical vibrations, celestial mechanics, electrical circuits, and biomechanical systems, where high-frequency oscillations and stiffness are prevalent. The method's robustness makes it particularly valuable for engineering simulations and scientific computing, offering a more efficient alternative to conventional predictor-corrector and linear multistep methods in modeling complex oscillatory phenomena.

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